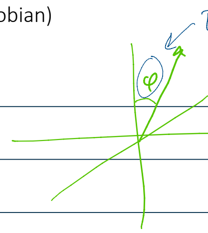


Definition:

To convert to spherical coordinates, we let

$x = \rho \cos\theta \sin\varphi$

$y = \rho \sin\theta \sin\varphi$

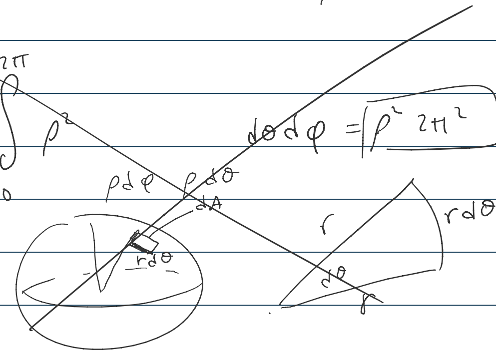
$z = \rho \cos\varphi$

$$\frac{4}{3}\pi r^3 = \text{Volume of sphere} = \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

And we need to add the integrating factor: $\boxed{\rho^2 \sin\varphi}$

https://mathinsight.org/spherical_coordinates

$$\int_0^{\pi}\int_0^{2\pi} \rho^2 \, d\theta\, d\varphi = \boxed{\rho^2\, 2\pi^2}$$

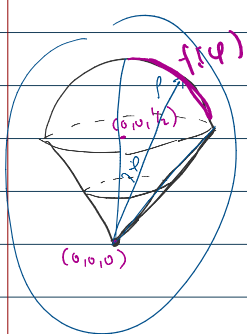


Ex: Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below $z = x^2+y^2+z^2$

$0 = x^2+y^2+z^2-z$

$= x^2+y^2+(z-\tfrac{1}{2})^2 - \tfrac{1}{4}$

$(\tfrac{1}{2})^2 = x^2+y^2+(z-1/2)^2$



Sphere => $\rho\cos\varphi = \rho^2$ => $\rho = \cos\varphi$

Cone => $\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}$

$= \rho\sin\varphi$ => $\cos\varphi = \sin\varphi$

$= \{\text{all } \rho,\varphi,\theta \text{ s.t. } \cos\varphi = \sin\varphi\}$

$= \{\ldots \text{ s.t. } \varphi = \pi/4\}$

Solid $= \{\text{all } \rho,\varphi,\theta \text{ s.t. } 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi/4,\ 0\le\rho\le\cos\varphi\}$

$f(\varphi)$

$\overbrace{(x(u,v),\, y(u,v))}^{F} : \mathbb{R}^2 \to \mathbb{R}^2$

DF 2x2 matrix

15.9: Change of coordinates (Jacobian)

$x = r\cos\theta \qquad y = r\sin\theta$

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2+y^2 \, dx\, dy = \int_0\int_0^{2\pi} r^2 \; r \; dr\, d\theta$$

In two dimensions:

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Jacobian

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Jacobian

$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$

General:

$$\int_{T(\Omega)} f(\vec{x})\,dV(\vec{x}) = \int_{\Omega} f(T(\vec{x}))\,|\det(D\,T)(\vec{x})|\,dV(\vec{x})$$

Note that in one variable, this is just normal u substitution

Ex: set up the change of coordinates for $x = r\cos\theta$, $y = r\sin\theta$

$x_r = \cos\theta \qquad y_r = \sin\theta$

$x_\theta = -r\sin\theta \qquad y_\theta = r\cos\theta$

$$J = \det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = x_r y_\theta - x_\theta y_r$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

Exercises:

1. Describe what the surface $\phi = \frac{\pi}{3}$ looks like
2. Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$ where E is enclosed by the sphere $x^2+y^2+z^2 = 9$ and the first octant
3. Set up the determinant to determine the integration constant for integrating in spherical coordinates
4. Show that a standard 1 variable u substitution follows from the change of variables formula
5. Find the Jacobian of the transformation $x = uv, y = \frac{u}{v}$
6. Compute $\iint_R x^2\,dA$ where R is the region bounded by the ellipse $9x^2+4y^2 = 36$, (use the sub $x = 2u, y = 3v$)

$J = \det\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = 6 \qquad 9\cdot(4)u^2 + 4\cdot 9v^2 = 36$

$u^2 + v^2 = 1$

$$\iint_R (2u)^2 \quad 6\,du\,dv$$

$$8 \iint 3 u^2 \, du \, dv$$

over
circle
radius 1

$$\left.\begin{aligned} \frac{x}{2} &= u = r\cos\theta \\ \frac{y}{3} &= v = r\sin\theta \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 2r\cos\theta \\ y &= 3r\sin\theta \end{aligned}$$